Lecture I From Materials to K-theory to Operator Algs (Meyer)

Topology
Analysis $>$ Quantum mechanics.

from electrons in crystalline materials to a vector bundle over $(S^1)^d$.

in quantum mechanics:

state $\sim$ unit vector ψ in a Hilbert space $\mathcal{H}$. $T \mapsto \langle \psi | T\psi \rangle$

observable = self-adjoint operator T on $\mathcal{H}$.

expectation value $\langle \psi | T(\psi) \rangle$

e.g. one particle moving in $\mathbb{R}^3$.

$$\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^2)$$

- Hamiltonian H: energy observable
  unitary time evolution:
  $\mathcal{U}_t : \mathcal{H} \to \mathcal{H}$, $\mathcal{U}_t$ (ψ at time t_0) $=$ (ψ at time t_0+t)

$$\mathcal{U}_t = \exp\left(-\frac{itH}{\hbar}\right)$$

e.g. $H = -c \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} + V$, $V : \mathbb{R}^3 \to \mathbb{R}$ potential

How do simplify the setup for electrons in a crystal?

One particle approximation: pretend that there is only one electron moving into a potential ("classical mechanics") generated by the other electrons and the atomic nuclei.

crystal: lattice $\Lambda \subseteq \mathbb{R}^3$, and $V(x+y) = V(x)$ $\forall y \in \Lambda$, $x \in \mathbb{R}$.

Neglect disorder from impurities and temperature $> 0K$ here.

unit cell C: $C + \Lambda = \mathbb{R}^3$ (disjoint copies)

$$L^2(\mathbb{R}^3) \cong L^2(C) \otimes \ell^2(\Lambda)$$

Next approximation: Replace $L^2(C)$ by a finite-dimentional subspace $\mathbb{C}^N$

$\rightsquigarrow$ self-adjoint operators $\mathcal{H}$ on $\ell^2(\Lambda) \otimes \mathbb{C}^N$.

$$\left(\Lambda \cong \underset{\underset{\mathbb{R}^d}{\cap \mid}}{\mathbb{Z}^d} \right)$$

H commutes with $S_x : \ell^2(\Lambda, \mathbb{C}^N) \circlearrowleft$

$$(S_x f)(y) = f(x+y)$$

$$H = (H_{x,y})_{x,y \in \mathbb{Z}^d}, \quad H_{x,y} \in M_N(\mathbb{C})$$

$$(Hf)(x) = \sum_{y \in \mathbb{Z}^d} H_{x,y}(f(y))$$

$\exists R > 0 : H_{x,y} = 0$ for $\|x-y\| \geq R$

$$\Rightarrow H = \sum_{\|x\| \leq R} S_x \otimes \tilde{H}_x, \quad \tilde{H}_x \in M_N(\mathbb{C})$$

$$\tilde{H}_{-x} = \tilde{H}_x^*$$

Fourier transf. $\ell^2(\mathbb{Z}^d) \cong L^2(\mathbb{T}^d)$ ← torus

H become pointwise application of a function $\mathbb{T}^d \xrightarrow{\hat{H}} M_N(\mathbb{C})$

$$(\hat{H} f)(k) = H(k)(f(k)) \quad \text{for } f \in L^2(\mathbb{T}^d, \mathbb{C}^N)$$

$\Rightarrow$ spectrum $= \bigcup_{i=1}^{N} [x_i, y_i]$ energy bound

insulator means:

"Fermi energy" not in $\sigma(H)$

then may form the spectral projection

$$\chi_{(-\infty, E]}(H) = P : \mathbb{T}^d \to \text{Projection on } \mathbb{C}^N$$

continuous

$\operatorname{im}(P_k)$ for $k \in \mathbb{T}^d$ is a vector bundle.

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Date **Meyer. Lecture II**

H hamiltonian $\longmapsto$ vector bundle $\xi \xrightarrow{\pi} \mathbb{T}^d \subseteq \mathbb{C}^d$

$E \notin \sigma(H)$

current

integer n $\xleftarrow{\text{Chern character}}$ $K^*(\mathbb{T}^d) \xrightarrow{ch} H^*(\mathbb{T}^d, \mathbb{R})$

$\Theta : \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \circlearrowleft$ antiunitary, e.g.

$\Theta(f(x)) = \overline{f(x)} \quad \forall x \in \mathbb{Z}^d$

$\Theta^2 = \mathrm{Id}, \quad H\Theta = \Theta H$

induces $\zeta : \xi \to \xi$ time-reversal symetry

$\zeta^2 = \mathrm{id}$.

$\pi \circ \zeta(x) = \overline{\pi(x)} \qquad x \in \xi$

$\zeta(\xi_z) = \xi_{\bar{z}}$

Atiyah: "real" K-theory, if you take $\mathbb{R} + i\mathbb{R} = \mathbb{C}$

then $KR^*(X \times \mathbb{R} \times i\mathbb{R}) \cong KR^*(X)$

but $KR^*(\mathbb{R}^8 \times X) \cong KR^*(X)$.

$H \longmapsto P_{(-\infty, E)}(H) \qquad P_1 \cdot C(\mathbb{T}^d, \mathbb{C}^N) \cong$

$\cong \Gamma(E_1)$

$E_1 \cong E_2 \implies \begin{pmatrix} P_1 & 0 \\ 0 & 0_0 \end{pmatrix} \underset{\text{homotopic}}{\sim} \begin{pmatrix} P_2 & 0 \\ 0 & 0_0 \end{pmatrix}$

We care about

$$P_1 \oplus \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & 0 \end{pmatrix} \underset{\text{homotopic}}{\sim} P_2 \oplus \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & 0 \end{pmatrix}$$

The stable isomorphism relation that defines $K^0(X)$ is also the right relation to define a <u>topological phase</u>.

How to choose $H: \ell^2(\mathbb{Z}^d, \mathbb{C}^N)$? for a given material? [Bernevig, Hughes, Zhang 2006]

First Choose N. (Example, $N=4$)

look at the symmetries of the crystal: H should be invariant under it.

Symmetries act on $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ by unitaries commuting with H: $\rightsquigarrow$ constraints on H.

Write down the most general $\sum_{0<|x|\text{ minimal}} S_x \otimes H_x$, $H_x \in M_4(\mathbb{C})$ obeying these constraints.

$k \in (\mathbb{R}/\mathbb{Z})^d$

Meyer Lecture III

- Which observable algebra A? Predicts $K_*(A)$, that is, which topological phases exist?

E.g. $A = \mathbb{B}(\underbrace{\ell^2(\mathbb{Z}^d, \mathbb{C}^N)}_{\mathcal{H}}) \ni H, \quad 0 \notin \sigma(H)$

$$P_{(-\infty,0]}(H) \in \mathbb{B}(\mathcal{H})$$

For periodic H, $\operatorname{rank} P_{(-\infty,0]}(H) = \infty$, all such projections are homotopic.

$\ulcorner$ $P \in \mathbb{B}(\mathcal{H})$ projection $\Rightarrow \exists\, \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$,

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \dim \mathcal{H}_0, \mathcal{H}_1 = \infty$$

$$\mathcal{H} = \mathcal{K}_0 \oplus \mathcal{K}_1, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\exists$ unitary: $\mathcal{H}_i \xrightarrow{u} \mathcal{K}_i, \quad i = 0, 1$

$$uPu^* = Q$$

$\exists\, (u_j)_{j \in [0,1]}$ norm continuous $u_0 = 1$, $u_1 = u_0$

$u_j P u_j^*$ homotopy $P \sim Q$ $\lrcorner$

We should use some properties of our Hamiltonian to define smaller A.
which properties?

- periodicity: A consists of operators commuting with translations S_x, $x \in \mathbb{Z}^d$
- locality: $H_{x,y} = 0$ for $\|x-y\| > R$, $x, y \in \mathbb{Z}^d$
  $\rightsquigarrow$ should take norm closure.

Last time: $\sum_{x \in \mathbb{Z}^d} S_x \otimes H_x$ finite / "norm-convergent"

position operators $(X_i f)(x) = x_i f(x)$, $x \in \mathbb{Z}^d$, $i = 1, \dots, d$, generate unitary representation of $\mathbb{T}^d$ on $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$
eigenspace for induced actions on $\mathbb{B}(\ell^2(\mathbb{Z}^d, \mathbb{C}^N))$

are $\{ H \in \mathbb{B}(\mathcal{H}) \mid H_{x,y} \neq 0 \Rightarrow x - y = z \}$
for $z \in \mathbb{Z}^d = \widehat{\mathbb{T}^d}$.
continuity for $\mathbb{T}^d$-action singles out $C(\mathbb{T}^d, M_N \mathbb{C}) \subseteq L^\infty(\mathbb{T}^d, M_N \mathbb{C})$.
To allow disorder, give up periodicity.
Locality remains.
Choose $A := \overline{\{ H \in \mathbb{B}(\mathcal{H}) \mid H_{x,y} = 0 \text{ for } \|x-y\| \geq R, \text{ some } R \}}^{\text{closure}} =$
$= \{ H \in \mathbb{B}(\mathcal{H}) \mid \mathbb{T}^d \ni t \mapsto \alpha_t(H) \text{ norm-continuous} \}$
$\alpha_t \in \mathrm{Aut}(\mathbb{B}(\mathcal{H}))$, $\alpha_t(H_{x,y}) = e^{it(x-y)} \cdot H_{x,y}$ $\quad x, y \in \mathbb{Z}^d, t \in \mathbb{T}^d$

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Date This A is the uniform Roe algebra of $\mathbb{Z}^d$ (Kubota)

Roe Algebra of $\mathbb{Z}^d$: Work on $\ell^2(\mathbb{Z}^d, \ell^2(\mathbb{N}))$

Besides locality as below, require local compactness:

$$H_{x,y} \in \mathbb{K}(\ell^2\mathbb{N}) \quad \forall x,y \in \mathbb{Z}^d$$

$$C^*_{Roe}(\mathbb{Z}^d) = \{ H \in \mathbb{B}(\ell^2(\mathbb{Z}^d, \ell^2(\mathbb{N}))) \mid H \text{ local, locally compact} \}$$

$$= \{ (H_{x,y}) \mid H_{x,y} \in \mathbb{K},\ H_{x,y} = 0 \text{ for } \|x-y\| \text{ large} \}$$

Or work on $L^2(\mathbb{R}^d)$;

$C^*_{Roe}(\mathbb{Z}^d)$ non-unital.

$$K_\ell(C^*_{Roe}(\mathbb{Z}^d)) = \begin{cases} \mathbb{Z} & (d-\ell) \text{ even} \\ 0 & (d-\ell) \text{ odd.} \end{cases}$$

Meyer; lecture IV

$C(\Omega) \rtimes \mathbb{Z}^d$ e.g. $\Omega = \prod_{\mathbb{Z}^d} [-1,1]$ - gives a kind of restricted disorder.

uniform Roe $\cong \ell^\infty(\mathbb{Z}^d) \rtimes \mathbb{Z}^d \otimes M_n \mathbb{C}$

$\Omega \ni \omega : \mathbb{Z}^d \to \Omega, \quad x \mapsto x \cdot \omega$, induces

$$C(\Omega) \rtimes \mathbb{Z}^d \hookrightarrow \underbrace{\ell^\infty(\mathbb{Z}^d)}_{C(\beta \mathbb{Z}^d)} \rtimes \mathbb{Z}^d$$

- Stone-Čech of $\mathbb{Z}^d$.

$$\text{Roe} \cong \ell^\infty(\mathbb{Z}^d, \mathbb{K}(\ell^2 \mathbb{N})) \rtimes \mathbb{Z}^d.$$

$K_0(\ell^\infty(\mathbb{Z}^d)) = \{ \text{bounded functions } \mathbb{Z}^d \to \mathbb{Z} \}$

$K_0(\ell^\infty(\mathbb{Z}^d, \mathbb{K})) = \{ \text{functions } \mathbb{Z}^d \to \mathbb{Z} \}$

$\Downarrow$

$$K_\ell(C^*_{Roe}\, \mathbb{Z}^d) \cong \begin{cases} \mathbb{Z} & d = \ell \\ 0 & d \neq \ell \end{cases}$$

$\ell^\infty(\mathbb{Z}^d) \rtimes \mathbb{Z}^d$ has traces, $C^*_{Roe}(\mathbb{Z}^d)$ not.

$C^*_{Roe}(X)$ defined $\forall X$ metric space.

$C^*_{Roe}(\mathbb{R}^d) \cong C^*_{Roe}(X)$ for $X \subseteq \mathbb{R}^d$ s.t. $\exists R > 0\ \forall y \in \mathbb{R}^d\ \exists x \in X: d(x,y) \leq R$.

magnetic twists do not change $C^*_{Roe}(X)$

$C(\mathbb{T}^d) \otimes \mathbb{K} \subseteq C^*_{Roe}(\mathbb{Z}^d)$ this is the subalgebra of periodic elements

induces map $K_*(C(\mathbb{T}^d) \otimes \mathbb{K}) \to K_*(C^*_{Roe}(\mathbb{Z}^d))$

what is it?

$$C_0(\mathbb{R}) \rightarrowtail C(\mathbb{T}) \underset{\curvearrowleft}{\xrightarrow{ev_1}} \mathbb{C} \Rightarrow K_*(C(\mathbb{T})) \cong K_*(C_0(\mathbb{R})) \oplus K_*(\mathbb{C})$$

$\Rightarrow C_0(\mathbb{R}) \oplus \mathbb{C} \sim C(\mathbb{T})$ in bivariant K-theory

$$\Rightarrow \underbrace{(C_0(\mathbb{R}) \oplus \mathbb{C})^{\otimes d}}_{\bigoplus_{j=0}^{d} \binom{d}{j} C_0(\mathbb{R}^j)} \sim C(\mathbb{T}^d) \Rightarrow K_*(C(\mathbb{T}^d)) \cong \bigoplus_{j=0}^{d} \binom{d}{j} K_*(C_0(\mathbb{R}^j))$$

$$K_*(C^*_{Roe}(\mathbb{N} \times \mathbb{Z}^{d-1})) \cong 0$$

$$\begin{tikzcd} C(\mathbb{T}^{d-1}) \arrow[r, phantom, "\subseteq"] \arrow[d] & C(\mathbb{T}^d) \arrow[d] \\ C^*_{Roe}(\mathbb{Z}^{d-1} \times \mathbb{N}) \arrow[r, phantom, "\subseteq"] & C^*_{Roe}(\mathbb{Z}^d) \end{tikzcd} \Rightarrow \begin{tikzcd} K_*(C(\mathbb{T}^{d-1})) \arrow[d, "\text{vanishes}"] \\ K_*(C^*_{Roe}(\mathbb{Z}^d)) \end{tikzcd}$$

Only top-dim. summand $C_0(\mathbb{R}^d)$ remains; it is the only one that does not come from some $C(\mathbb{T}^{d-1}) \subseteq C(\mathbb{T}^d)$. The map

$$K_*(C_0(\mathbb{R}^d)) \to K_*(C(\mathbb{T}^d)) \to K_*(C^*_{Roe}(\mathbb{Z}^d)) \cong K_{*+d}(\mathbb{C})$$

is the suspension isomorphism.

$$\begin{tikzcd} K_*(C_0(\mathbb{R}^d)) \arrow[r, "\cong"] \arrow[d] & \mathbb{Z} \\ K_*(C(\mathbb{T}^d)) \arrow[ur, "\text{fundamental class}"'] & \end{tikzcd}$$

$\xrightarrow{\text{Fourier trans.}} \ell^2(\mathbb{Z}^d, \mathbb{C}^\ell)$

$\sum x_j \partial_j$

$L^2(\mathbb{T}^d, \text{Spinor})$, diff. operator

So, we see exactly the strong topo.
phase when we go from $C(\mathbb{T}^d)$ to $C^*_{Roe}(\mathbb{Z}^d)$