

Lecture I | From Materials to K -theory
to Operator Algs (Meyer)

Topology > Quantum mechanics.
Analysis

from electrons in crystalline
materials to a vector bundle over $(S^1)^d$

in quantum mechanics:

state is unit vector ψ in a
Hilbert space. $\mathcal{H} \cdot |T\rangle \langle \psi| T \psi\rangle$

observable ~~is~~ = self-adjoint
operator T on \mathcal{H} .

expectation value $\langle \psi | T(\psi) \rangle$

e.g. one particle moving in \mathbb{R}^3 .

$$\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^2)$$

• Hamiltonian H : energy observable
unitary time evolution:

$$U_t: \mathcal{H} \rightarrow \mathcal{H}, \quad U_t (\psi \text{ at time } t_0) \\ = (\psi \text{ at time } t_0 + t)$$

$$U_t = \exp\left(-\frac{itH}{\hbar}\right)$$

e.g. $H = -\hbar^2 \sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2} + V, \quad V: \mathbb{R}^3 \rightarrow \mathbb{R}$ potential

How do simplify the setup for electrons in a crystal?

One particle approximation: pretend that there is only one electron moving into a potential ("classical mechanics") generated by the other electrons and the atomic nuclei.

crystal: lattice $\Lambda \subseteq \mathbb{R}^3$, and
 $V(x+y) = V(x) \quad \forall y \in \Lambda, x \in \mathbb{R}^3$.

Neglect disorder from impurities and temperature $\gg 0$ K here.

unit cell C : $C+1 = \mathbb{R}^3$ (disjoint copies)

$$L^2(\mathbb{R}^3) \cong L^2(C) \otimes \ell^2(1)$$

Next approximation: Replace $L^2(C)$ by a finite-dimensional subspace \mathbb{C}^N

\leadsto self-adjoint operators H on $\ell^2(1) \otimes \mathbb{C}^N$
($1 \cong \sum_{n=1}^d \mathbb{R}^d$)

H commutes with $S_x: \ell^2(1, \mathbb{C}^N) \rightarrow \ell^2(1, \mathbb{C}^N)$
($S_x f$)(y) = $f(x+y)$

$H = (H_{x,y})_{x,y \in \mathbb{Z}^d}$, $H_{x,y} \in M_N(\mathbb{C})$

$$(Hf)(x) = \sum_{y \in \mathbb{Z}^d} H_{x,y} (f(y))$$

$\exists R > 0$: $H_{x,y} = 0$ for $\|x-y\| \geq R$

$\Rightarrow H = \sum_{\|x\| \in R} S_x \otimes \tilde{H}_x$, $\tilde{H}_x \in M_N(\mathbb{C})$

$$\tilde{H}_{-x} = \tilde{H}_x^*$$

Fourier transf. $L^2(\mathbb{Z}^d) \cong L^2(\mathbb{T}^d)$ ^{forms}

H becomes pointwise application of a function $\mathbb{T}^d \xrightarrow{\hat{H}} M_N(\mathbb{C})$

$$(\hat{H} f)(k) = H(k)(f(k)) \quad \text{for } f \in L^2(\mathbb{T}^d, \mathbb{C}^N)$$

$$\Rightarrow \text{spectrum} = \bigcup_{i=1}^N [x_i, y_i] \quad \text{energy bands}$$

insulator means:

"Fermi energy" not in $\sigma(H)$

then may form the spectral projection

$$\chi_{(-\infty, E]}(H) = P : \mathbb{T}^d \rightarrow \text{Projection on } \mathbb{C}^N$$

↑
continuous

$\text{im}(P_k)$ for $k \in \mathbb{T}^d$ is a vector bundle.

Meyer. Lecture II

H hamiltonian \mapsto vector bundle $\xi \xrightarrow{\pi} \mathbb{T}^d \subseteq \mathbb{C}^d$
 $E \notin \sigma(H)$
 \downarrow Chern character
 current integer n $K^*(\mathbb{T}^d) \xrightarrow{ch} H^*(\mathbb{T}^d, \mathbb{R})$

$\Theta : \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \ni$ antiunitary, e.g.
 $\Theta(f(x)) = \overline{f(x)}$ $\forall x \in \mathbb{Z}^d$
 $\Theta^2 = Id, H\Theta = \Theta H$

induces $\zeta : \xi \rightarrow \xi$ time-reversal symmetry
 $\zeta^2 = id$

$$\pi \circ \zeta(x) = \overline{\pi(x)} \quad x \in \xi$$

$$\zeta(\xi_{\bar{z}}) = \xi_z$$

Atiyah: "real" K -theory, if you take $\mathbb{R} + i\mathbb{R} = \mathbb{C}$
 then $K\mathbb{R}^*(X + \mathbb{R} + i\mathbb{R}) \cong K\mathbb{R}^*(X)$
 but $K\mathbb{R}^*(\mathbb{R}^8 + X) \cong K\mathbb{R}^*(X)$

$$H \mapsto P_{(-\infty, \epsilon)}(H) \quad P_1 \cdot C(\mathbb{T}^d, \mathbb{C}^N) \cong \Gamma(E_1)$$

$$E_1 \cong E_2 \Rightarrow \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix} \sim_{\text{homotopic}} \begin{pmatrix} P_2 & 0 \\ 0 & 0 \end{pmatrix}$$

We care about

$$P_1 \oplus \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \underset{\text{homotopic}}{\sim} P_2 \oplus \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

The stable isomorphism relation that defines $K^0(X)$ is also the right relation to define a topological phase.

How to choose $H: \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \rightarrow \mathbb{C}^N$ for a given material? [Berezin, Hughes, Zhang 2006]

First choose N . (Example, $N=4$)

look at the symmetries of the crystal:

H should be invariant under it.

Symmetries act on $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ by unitaries commutably with H . \implies constraints on H .

Write down the most general

$$\sum_{0 < |x| \leq \text{minimal}} S_x \oplus H_x, \quad H_x \in M_N(\mathbb{C}) \text{ obeying these constraints.}$$

$$k \in (\mathbb{R}/\mathbb{Z})^d$$

Meyer Lecture III

• Which observable algebra A ?

Predicts $K_*(A)$, that is, which topological phases exist?

E.g. $A = \mathcal{B}(\ell^2(\mathbb{Z}^d, \mathbb{C}^N)) \ni H, 0 \notin \sigma(H)$

$$P_{(-\infty, 0]}(H) \in \mathcal{B}(\mathcal{H})$$

For periodic H , $\text{rank } P_{(-\infty, 0]}(H) = \infty$,
all such projections are homotopic.

\top $P \in \mathcal{B}(\mathcal{H})$ projection $\Rightarrow \exists \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$,

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \dim \mathcal{H}_0, \mathcal{H}_1 = \infty$$

$$\mathcal{H} = \mathcal{K}_0 \oplus \mathcal{K}_1, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

\exists unitary: $\mathcal{H}_i \xrightarrow{U} \mathcal{K}_i, i=0,1$
 $U P U^* = Q$

$\exists (U_j)_{j \in [0,1]}$ norm continuous $U_0 = 1, U_1 = U_0$
 $U_j P U_j^*$ homotopy P to Q

We should use some properties of our Hamiltonian to define smaller \mathcal{A} .
which properties?

◦ periodicity: \mathcal{A} consists of operators commuting with translations

$$S_x, x \in \mathbb{Z}^d$$

◦ locality: $H_{x,y} = 0$ for $\|x-y\| > R, x, y \in \mathbb{Z}^d$
→ should take norm closure.

Last time: $\sum_{x \in \mathbb{Z}^d} S_x \otimes H_x$ finite / "norm-convergent"

position operators $(x_i \uparrow)(x) = x_i, x \in \mathbb{Z}^d, i=1, \dots, d$, generate unitary

representation of \mathbb{T}^d on $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$
eigenspace for induced actions on $B(\ell^2(\mathbb{Z}^d, \mathbb{C}^N))$

are $\{H \in B(\mathcal{H}) \mid H_{x,y} \neq 0 \Rightarrow x-y = z\}$
for $z \in \mathbb{Z}^d = \widehat{\mathbb{T}^d}$

continuity for \mathbb{T}^d -action singles out $C(\mathbb{T}^d, M_N \mathbb{C}) \in L^\infty(\mathbb{T}^d, M_N \mathbb{C})$.

To allow disorder, give up periodicity.
Locality remains.

Choose $\mathcal{A} := \{H \in B(\mathcal{H}) \mid H_{x,y} = 0 \text{ for } \|x-y\| \geq R, \text{ some } R\}$ closure =

$= \{H \in B(\mathcal{H}) \mid \mathbb{T}^d \ni t \mapsto \alpha_t(H), \text{ norm-continuous}\}$
 $\alpha_t \in \text{Aut}(B(\mathcal{H})), \alpha_t(H_{x,y}) = e^{it(x-y)} H_{x,y}$ $x, y \in \mathbb{Z}^d$

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This A is the uniform Roe algebra of \mathbb{Z}^d (Kubota)

Roe Algebra of \mathbb{Z}^d : Work on $\ell^2(\mathbb{Z}^d, \ell^2(N))$
 besides locality as below, require local compactness:

$$M_{x,y} \in \mathbb{K}(\ell^2(N)) \quad \forall x,y \in \mathbb{Z}^d$$

$$C_{Roe}^*(\mathbb{Z}^d) = \left\{ M \in \mathcal{B}(\ell^2(\mathbb{Z}^d, \ell^2(N))) \mid \begin{array}{l} M \text{ local, locally compact} \end{array} \right\}$$

$$= \left\{ (M_{x,y}) \mid \begin{array}{l} M_{x,y} \in \mathbb{K}, \quad M_{x,y} = 0 \\ \text{for } \|x-y\| \text{ large} \end{array} \right\}$$

Or work on $L^2(\mathbb{R}^d)$:

$C_{Roe}^*(\mathbb{Z}^d)$ non-unital.

$$K_e(C_{Roe}^*(\mathbb{Z}^d)) = \begin{cases} \mathbb{Z} & (d-e) \text{ even} \\ 0 & (d-e) \text{ odd.} \end{cases}$$

Meyer; Lecture IV

$C(\Omega) \rtimes \mathbb{Z}^d$ e.g. $\Omega = \prod_{\mathbb{Z}^d} [-1, 1]$ - gives a

kind of restricted disorder.

uniform Roe $\cong C^\infty(\mathbb{Z}^d) \rtimes \mathbb{Z}^d \otimes M_n \mathbb{C}$

$\Omega \ni \omega : \mathbb{Z}^d \rightarrow \Omega, x \mapsto x \cdot \omega$, induces

$$C(\Omega) \rtimes \mathbb{Z}^d \hookrightarrow C^\infty(\mathbb{Z}^d) \rtimes \mathbb{Z}^d$$

$C(\beta\mathbb{Z}^d)$ - Stone-Cech of \mathbb{Z}^d .

$$\text{Roe} \cong C^\infty(\mathbb{Z}^d, K(C^*(\mathbb{Z}^d))) \rtimes \mathbb{Z}^d$$

$$K_0(C^\infty(\mathbb{Z}^d)) = \left\{ \text{bounded functions } \mathbb{Z}^d \rightarrow \mathbb{Z} \right\}$$

$$K_0(C^\infty(\mathbb{Z}^d, K)) = \left\{ \text{functions } \mathbb{Z}^d \rightarrow K \right\}$$

$$\Downarrow$$

$$K_e(C_{\text{Roe}}^* \mathbb{Z}^d) \cong \begin{cases} \mathbb{Z} & d=l \\ 0 & d \neq l \end{cases}$$

$C^\infty(\mathbb{Z}^d) \rtimes \mathbb{Z}^d$ has traces, $C_{\text{Roe}}^*(\mathbb{Z}^d)$ not.

$C_{\text{Roe}}^*(X)$ defined $\forall X$ metric space.

$$C_{\text{Roe}}^*(\mathbb{R}^d) \cong C_{\text{Roe}}^*(X) \text{ for } X \subseteq \mathbb{R}^d \text{ s.t.}$$

$$\exists R > 0 \forall y \in \mathbb{R}^d \exists x \in X:$$

$$d(x, y) \leq R.$$

magnetic twists do not change $C_{\text{Roe}}^*(X)$

$C(\mathbb{T}^d) \otimes \mathbb{K} \subseteq C_{\text{Roe}}^*(\mathbb{Z}^d)$ this is the

subalgebra of periodic elements

induces map $K_* (C(\mathbb{T}^d) \otimes \mathbb{K}) \rightarrow K_* (C_{\text{Roe}}^*(\mathbb{Z}^d))$
 what is it?

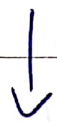
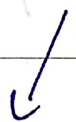
$$C_0(\mathbb{R}) \xrightarrow{\text{ev}_1} C(\mathbb{T}) \xrightarrow{\cong} \mathbb{C} \Rightarrow K_* (C(\mathbb{T})) \cong K_* (C_0(\mathbb{R})) \oplus K_* (\mathbb{C})$$

$\Rightarrow C_0(\mathbb{R}) \oplus \mathbb{C} \sim C(\mathbb{T})$ in bivariant K -theory

$$\Rightarrow \underbrace{(C_0(\mathbb{R}) \oplus \mathbb{C})^{\oplus d}}_{\bigoplus_{j=0}^d C_0(\mathbb{R}^j)} \sim C(\mathbb{T}^d) \Rightarrow K_* (C(\mathbb{T}^d)) \cong \bigoplus_{j=0}^d K_* (C_0(\mathbb{R}^j))$$

$$K_* (C_{\text{Roe}}^*(N \times \mathbb{Z}^{d-1})) \cong 0$$

$$C(\mathbb{T}^{d-1}) \subseteq C(\mathbb{T}^d)$$



$\Rightarrow K_* (C(\mathbb{T}^{d-1}))$
 \downarrow vanishes

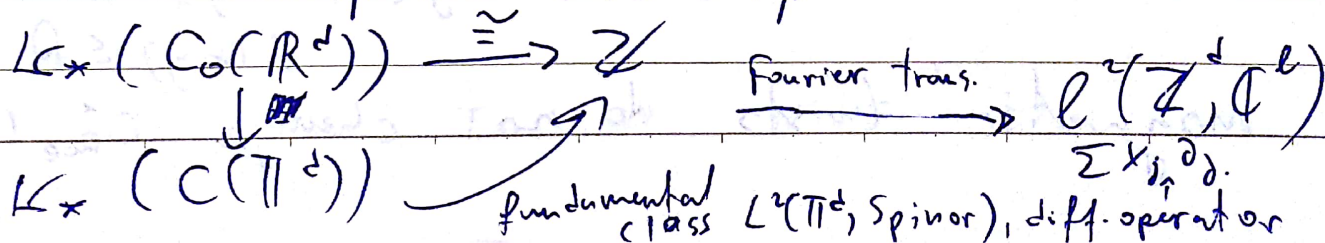
$$C_{\text{Roe}}^*(\mathbb{Z}^d \times N) \subseteq C_{\text{Roe}}^*(\mathbb{Z}^d)$$

$$K_* (C_{\text{Roe}}^*(\mathbb{Z}^d))$$

Only top-dim. summand $C_0(\mathbb{R}^d)$ remains; it is the only one that does not come from some $C(\mathbb{T}^{d-1}) \subseteq C(\mathbb{T}^d)$. The map

$$K_* (C_0(\mathbb{R}^d)) \rightarrow K_* (C(\mathbb{T}^d)) \rightarrow K_* (C_{\text{Roe}}^*(\mathbb{Z}^d)) \cong \cong K_{d-2}(\mathbb{C})$$

is the suspension isomorphism.



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So, we see exactly the strong topological phase when we go from $C(\mathbb{T}^d)$ to $C_{\text{free}}^*(\mathbb{Z}^d)$