Clifford-Fourier transforms and wavelets

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One-dimensional harmonic analysis

Single-channel signals

$$f : I \subset \mathbb{R} \to \mathbb{R}$$

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- Short-time Fourier transform, continuous and discrete wavelet transform, etc.

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- $P^{\pm}$ bounded projections: $\left[\dfrac{1}{2}\left(1 \pm \dfrac{\xi}{|\xi|}\right)\right]^2 = \dfrac{1}{2}\left(1 \pm \dfrac{\xi}{|\xi|}\right)$
- $P^{\pm}$ orthogonal projections: $\dfrac{1}{2}\left(1 + \dfrac{\xi}{|\xi|}\right)\dfrac{1}{2}\left(1 - \dfrac{\xi}{|\xi|}\right) = 0$

Hardy spaces: $L^2 = H^2_+ \oplus H^2_-$ where $H^2_{\pm} = P^{\pm}(L^2)$

One-dimensional complex analysis

- Singular integrals: $P^{\pm} = \frac{1}{2}(I + i\mathcal{H})$ where $\mathcal{H}$ is the Hilbert transform

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- Analytic signal:

$$f_a(t) = f(t) + i\mathcal{H}f(t) = |f_a(t)|e^{i\theta(t)} = 2P^{+}f(t)$$

Local amplitude $|f_a(t)|$; local phase $\theta(t)$

Example: $f(t) = e^{-\pi t^2}\cos t \longrightarrow f_a(t) = e^{-\pi t^2} e^{it}$
Local amplitude $e^{-\pi t^2}$; Local phase $\theta(t) = t$.

Multichannel signals

Our treatment of signals $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is generally ad hoc.

Example: $n = 2$, $m = 1$: grayscale images

Tensor product constructions – Fourier analysis (convolution theorem etc) ok, but complex analysis not so good:

$$\hat{f}(\xi_1, \xi_2) = \mathcal{F}_2 \mathcal{F}_1 f(\xi_1, \xi_2)$$
$$\mathcal{H} f(x_1, x_2) = \mathcal{H}_2 \mathcal{H}_1 f(x_1, x_2)$$

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Example: $n = 2$, $m = 3$: colour images

$$f(\mathbf{x}) = (R(\mathbf{x}), G(\mathbf{x}), B(\mathbf{x}))$$

Even Fourier analysis breaks down here:

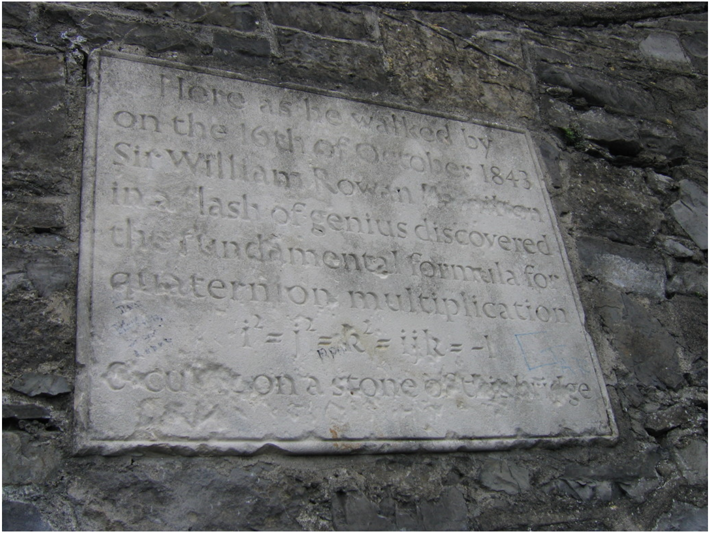
- $\hat{f}(\mathbf{y}) = \int_{\mathbb{R}^2} e^{-2\pi i \langle \mathbf{x}, \mathbf{y} \rangle} f(\mathbf{x})\, d\mathbf{x} = (\hat{R}(\mathbf{y}), \hat{G}(\mathbf{y}), \hat{B}(\mathbf{y}))$
- $f * g(\mathbf{x}) = \int_{\mathbf{R}^2} f(\mathbf{t}) g(\mathbf{x} - \mathbf{t})\, d\mathbf{t}$ not defined

Wish list

We want to

- Inject multichannel signals into an algebra that allows products of functions;
- With this algebraic structure, define a Fourier-type transform which maintains the useful covariances of the classical Fourier transform
- Build signal analysis and processing tools (wavelets etc) around the Fourier transform
- Build signal analytic tools analogous to the analytic signal for extracting local amplitude and phase information

Clifford algebra



Clifford algebra

- $\{e_1, e_2, \ldots, e_d\}$ an orthonormal basis for $\mathbb{R}^d$. Imbed $\mathbb{R}^d$ into a 2^d-dimensional associative Clifford algebra $\mathbb{R}_d$

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$$e_{\{j_1, j_2, \ldots, j_\ell\}} = e_{j_1} e_{j_2} \cdots e_{j_\ell}$$

$$e_\emptyset = e_0 = 1 \quad \text{(identity)}, \quad e_j^2 = -1$$

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- $\mathbb{R}_d = \{\sum_A x_A e_A; x_A \in \mathbb{R}\} = \Lambda_0 \oplus \Lambda_1 \oplus \cdots \oplus \Lambda_d = \Lambda_e \oplus \Lambda_o$
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- $\mathbb{C}_d = \{\sum_A z_A e_A; z_A \in \mathbb{C}\}$
- If $x \sum_{j=1}^d x_j e_j$, $y = \sum_{j=1}^d y_j e_j$ are vectors, then

$$x^2 = -|x|^2 \text{ and } xy = -\langle x, y \rangle + x \wedge y \in \Lambda_0 \oplus \Lambda_2$$

Examples

- $m = 1$, basis $\{e_0, e_1\}$, elements
  $u = a_0 + a_1 e_1, \quad v = b_0 + b_1 e_1$
- multiplication: $uv = a_0 b_0 - a_1 b_1 + (a_1 b_0 + b_0 a_1) e_1$, i.e.,
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- $m = 2$, basis $\{e_0, e_1, e_2, e_3 = e_{12} = e_1 e_2\}$, elements
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$$e_1^2 = e_2^2 = e_3^2 = -1, \; e_1 e_2 = e_3, \; e_3 e_1 = e_2, \; e_2 e_3 = e_1$$

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- $d = 3$, basis $\{e_0, e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123}\}$

Dirac operator

- We consider functions $f : \mathbb{R}^d \to \mathbb{R}_d$, i.e., $f : \mathbb{R} \to \mathbb{C}$, $f : \mathbb{R}^2 \to \mathbb{H}$, $f : \mathbb{R}^3 \to \mathbb{R}_3$.

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- We say f is (left) monogenic on Ω if $Df = 0$ (or $\partial f = 0$)
- $f : \mathbb{R}^{1+1} \to \mathbb{R}_1 = \mathbb{C}$, $f(x, y) = u(x, y) + e_1 v(x, y)$

$$\partial f = \frac{\partial f}{\partial x} + e_1 \frac{\partial f}{\partial y} = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + e_1 \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

So monogenicity $\equiv$ complex analyticity

Why Dirac operators?

- $H, E : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ vectorfields

$$H = H_1 e_1 + H_2 e_2 + H_3 e_3; \quad E = E_1 e_{23} + E_2 e_{31} + E_3 e_{12}$$

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- Dirac operators factorize the Laplacian and the Helmholtz operator:

$$D^2 = -\Delta; \qquad (D + ik)(D - ik) = -\Delta^2 + k^2$$

- Monogenic functions and Dirac operators play a fundamental role in electromagnetic/acoustic scattering theory.

Hypercomplex function theory

- Ω a domain in $\mathbb{R}^n$ with Lipschitz boundary, f left monogenic in Ω, $n(x)$ the outward pointing normal to Ω at $x \in \partial\Omega$ and $G(x) = \frac{x}{|x|^{n+1}}$. Then G is left and right monogenic on $\mathbb{R}^n \setminus \{0\}$ and

$$\frac{1}{\omega_n} \int_{\partial\Omega} G(x-y)n(y)f(y)\, d\sigma(y) = f(x)$$

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- Also have analogues of Liouville's theorem, mean-value theorem, Taylor theorem
- The product of monogenic functions is in general not monogenic!

Basic Operators of Clifford Analysis

Angular momentum operators: $\mathcal{L}_{ij} = x_i \partial_j - x_j \partial_i \quad (1 \leq i, j \leq d)$

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Commutation Relation: $[D, Q] = 2\Gamma - dI$

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Clifford-Hermite operators:

$$\mathcal{H}_d^+ = (D+Q)(D-Q) = -\Delta + |x|^2 + \Gamma - dI = \mathcal{H}_d + (\Gamma - (d/2)I)$$
$$\mathcal{H}_d^- = (D-Q)(D+Q) = -\Delta + |x|^2 - \Gamma + dI = \mathcal{H}_d - (\Gamma - (d/2)I)$$

Quaternionic Fourier transform for colour images

Quaternionic FT's: (Ell, Sangwine,...)

$$\mathcal{F}_1 f(u) = \int_{\mathbb{R}^2} f(x) e^{-2\pi e_1 u_1 x_1} e^{-2\pi e_2 u_2 x_2} \, dx$$

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- no convolution theorem
- lacking covariances

Fractional Clifford-Fourier transform (frCFT)

Classical fractional Fourier transform (frFT)

$$\mathcal{F}_t f(y) = e^{it\mathcal{H}_d} f(y) = \int_{\mathbb{R}^d} K_t(y,x) f(x)\, dx \qquad (t \in \mathbb{R})$$

with $\mathcal{F}_{\pi/2} = \mathcal{F}$ and

$$K_t(x,y) = \sqrt{\frac{-ie^{it}\csc t}{2\pi}} \exp(-i(\csc t)xy + i(\cot t)(|x|^2 + |y|^2)/2).$$

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More generally: $\mathcal{F}_t^{\pm} = \exp(-it\mathcal{H}_d^{\pm})$ (frCFT)

Fractional Clifford-Fourier transform

frCFT kernel:

$$\begin{aligned}\mathcal{F}_t^{\pm} &= \exp(-it(\mathcal{H}_d \pm (\Gamma - d/2))\\ &= \exp(\mp it(\Gamma - d/2))\exp(-it\mathcal{H}_d)\\ &= \exp(\mp it(\Gamma - d/2))\mathcal{F}_t.\end{aligned}$$

$$\boxed{C_t^{\pm}(x,y) = \exp(\pm itd/2)\exp(\mp it\Gamma_y)K_t(x,y)}$$

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Note: Terms in $\Gamma = -\sum_{1 \leq i < j \leq d} e_i e_j \mathcal{L}_{ij}$ not not commute.

Initial value problems

Theorem (Craddock, H. (JFAA 2013))

f is scalar-valued then $\exp(it\Gamma)f(x) = u(x,t) + \Gamma w(x,t)$ *with u, w scalar-valued satisfying the initial value problems*

$$\frac{\partial^2 u}{\partial t^2} + i(d-2)\frac{\partial u}{\partial t} = |x|^2 \Delta_T u \quad (x \in \mathbb{R}^d,\ t > 0)$$

$$u(x,0) = f(x) \quad (x \in \mathbb{R}^d)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \quad (x \in \mathbb{R}^d)$$

Mean-value solutions

Theorem (Gonzalez, Zhang (Contemp. Math. 2006))

d even:

$$u(x,t) = c_d \left[\frac{d}{dt} \left(- \frac{d}{d(\cos t)} \right)^{(d-4)/2} ((\sin t)^{d-3} M^t f(x)) \right]$$

d odd:

$$u(x,t) = c_d \frac{d}{dt} \int_0^t \frac{\left[\left(- \frac{d}{d(\cos s)} \right)^{(d-3)/2} (\sin s)^{d-3} M^s f(x) \right]}{\sqrt{\cos s - \cos t}} \sin s \, ds$$

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$d = 3$:

$$\boxed{u(x,t) = c \frac{d}{dt} \int_0^t \frac{M^s f(x)}{\sqrt{\cos s - \cos t}} \sin s \, ds}$$

$d = 2$: frCFT kernel

When $d = 2$, the IVP simplifies:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \theta^2} \quad (x = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2,\ \ t > 0)$$
$$u(x,0) = f(x) \quad (x \in \mathbb{R}^2)$$
$$u_t\big|_{t=0} = 0 \quad (x \in \mathbb{R}^2)$$

has d'Alembert solution: $u = \dfrac{f(\theta + t) + f(\theta - t)}{2}$

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$$C_t^{(2)}(x,y) = \frac{-ie^{it}}{2\pi \sin t} e^{(i/2)\cot t |x-y|^2} e^{-x \wedge y}$$

$$\boxed{\mathcal{F}^+ f(y) = \int_{\mathbb{R}^2} e^{y \wedge x} f(x)\, dx = \int_{\mathbb{R}^2} e^{e_1 e_2 (x_2 y_1 - x_1 y_2)} f(x)\, dx}$$

See also: Brackx, De Schepper, Sommen: J Math Imag Vis (2006)

$d > 2$: Separation of variables

$u = u(x, y, t) = u(z, \omega, t)$ with $z = |x||y|$, $\omega = \langle x, y\rangle / z$.

$f(x) = F(\langle x, y\rangle) = F(|x||y|\omega)$.

$$u = \sum_{\ell=0}^{\infty} \left(\frac{(\ell + d - 2)e^{i\ell t} + \ell e^{i(2-d-\ell)t}}{2\ell + d - 2} \right) \times \left(\int_{-1}^{1} F(s) P_\ell^d(s)(1 - s^2)^{(d-3)/2}\, ds \right) N(d, \ell) P_\ell^d(\omega)$$

$d = 4$: solution of the IVP

d'Alembert-like solution:

$$z = |x||y|,\ f(x) = F(\langle x, y\rangle) = F(z\cos\theta)$$

$$\begin{aligned} u &= u(z,\theta,t) \\ &= \frac{e^{-it}}{2\sin\theta}\Big[\sin(\theta+t)F(z\cos(\theta+t)) + \sin(\theta-t)F(z\cos(\theta-t)) \\ &\qquad + i\int_{\cos(\theta+t)}^{\cos(\theta-t)} F(zs)\,ds\Big] \end{aligned}$$

$d = 4$: frCFT kernel

Theorem (Craddock, H. (JFAA 2013))

$C_t^{(4)}(x,y) = u + v$, $u \in \Lambda_0$, $v \in \Lambda_2$ *and*

$$u = -\frac{e^{3it}}{4\pi^2 \sin t} e^{(i/2)\cot t|x-y|^2}\left[\cot t \cos|x \wedge y| + i\frac{\langle x, y\rangle}{|x \wedge y|}\sin|x \wedge y| + i\frac{\sin|x \wedge y|}{|x \wedge y|}\right]$$

$$v = \cdots$$

Method of ascent

Theorem (Craddock, H. (JFAA 2013))

Let $d > 2$, $g \in C^1[-1,1]$ and G an antiderivative of g, then

$$\boxed{u^e_{d+2}(g) = \frac{e^{-it}}{z}\frac{\partial u^e_d(G)}{\partial \omega}; \quad u^o_{d+2}(g) = \frac{d}{d-2}\frac{e^{-it}}{z}\frac{\partial u^o_d(G)}{\partial \omega}}$$

Unexpected connections

$$e^{it\Gamma_x}(\langle x, y\rangle^m) = \sum_{\ell=0}^{m} c_\ell^{(m)}(t)\langle x, y\rangle^{m-\ell}(x \wedge y)^\ell$$

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$$\frac{d}{dt}\mathbf{c}^{(m)}(t) = iA^{(m)}\mathbf{c}^{(m)}; \quad \mathbf{c}^{(m)}(0) = e_0 \Rightarrow \mathbf{c}^{(m)}(t) = e^{itA^{(m)}}e_0.$$

Unexpected connections

$$A^{(2N)} = \begin{pmatrix} 0 & 2N & 0 & 0 & \dots & \dots & \dots \\ d-1 & 2-d & 2N-1 & 0 & \dots & \dots & \dots \\ 0 & 2 & 0 & 2N-2 & \dots & \dots & \dots \\ 0 & 0 & d+1 & 2-d & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 2-d & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 2N & 0 \end{pmatrix}$$

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$A^{(m)}$ associated with the recurrence relations for the dual -1 Hahn polynomials. Eigenvectors are values of these polynomials

- Orthogonality relations used to compute the exponentials
- Generating functions to compute the resulting sums in closed form

Properties of the CFT

- Eigenfunction property: $D_x C_d^+(x,y) = C_d^-(x,y)y$
- Mapping properties: $\mathcal{F}^+ : L^1 \to L^\infty,\ \mathcal{S} \to \mathcal{S},\ L^2 \to L^2$
- Plancherel: $\int_{\mathbb{R}^2} \overline{f(x)} g(x)\, dx = (f,g) = (\mathcal{F}_d^+ f, \mathcal{F}_d^+ g)$
- Inversion: $(\mathcal{F}_d^+)^2 = I$
- Covariances:

$$\mathcal{F}_2^+ \tau_h = e^{y \wedge h} \mathcal{F}_2^+; \qquad \mathcal{F}_2^+ (e^{x \wedge h} f) = \tau_h \mathcal{F}_2^+; \qquad \rho \mathcal{F}_2^+ = \mathcal{F}_2^+ \rho^{-1}$$

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Theorem

Let $\sigma \in SO(d)$ and $z = z_\sigma \in Spin(d)$ such that $\sigma(x) = zx\overline{z}$ for all $x \in \Lambda_1$. Let $S_z f(x) = \overline{z} f(zx\overline{z})z$. Then

$$\mathcal{F}_d^+ S_z = S_z \mathcal{F}_d^+$$

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cf. classical FT: $\mathcal{F}R_\sigma = R_\sigma^{-1}\mathcal{F}$

$d = 2$: Quaternionic signal processing

Definition

An parity matrix is one of the form $A(\xi) = \begin{pmatrix} s(\xi) & v(\xi) \\ v(-\xi) & s(-\xi) \end{pmatrix}$ with $s : \mathbb{R}^d \to \Lambda_e$ and $v : \mathbb{R}^d \to \Lambda_o$. $A(\xi)^* = \overline{A}(\xi)^T$.

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Given $f : \mathbb{R}^d \to \mathbb{R}_d$, its associated parity matrix is

$$[f(x)] = \begin{pmatrix} f_e(x) & f_o(x) \\ f_o(-x) & f_e(-x) \end{pmatrix}$$

where $f(x) = f_e(x) + f_o(x)$ and $f_e : \mathbb{R}^d \to \Lambda_e$, $f_o : \mathbb{R}^d \to \Lambda_o$.

Convolution theorem ($d = 2$)

Convolution-filtering: $f : \mathbb{R}^2 \to \mathbb{H}$, $\hat{f} = \mathcal{F}_2^+ f$

$$\widehat{f * g}(y) = \int e^{y \wedge x} \int f(x - t) g(t) \, dt \, dx$$

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Theorem (H, Morris (2013))

$$\boxed{\widehat{f * g}(y) \neq \hat{f}(y)\hat{g}(y) \text{ but } [\widehat{f * g}(y)] = [\hat{f}(y)][\hat{g}(y)]}$$

Translation-invariance

Theorem (H, Morris (2013))

$T : L^2(\mathbb{R}^2, \mathbb{H}) \to L^2(\mathbb{R}^2, \mathbb{H})$ is bounded, right $\mathbb{H}$-linear and translation-invariant if and only if there exists a bounded parity matrix $A(\xi)$ such that

$$[\widehat{Tf}(\xi)] = A(\xi)[\hat{f}(\xi)].$$

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Theorem (H, Morris (2013))

$X \subset L^2(\mathbb{R}^2, \mathbb{H})$ is a closed translation-invariant right $\mathbb{H}$-linear submodule of $L^2(\mathbb{R}^2, \mathbb{H})$ if and only if there exists an idempotent self-adjoint parity matrix $A(\xi)$ such that $[\hat{f}(\xi)] = A(\xi)[\hat{f}(\xi)]$ for all $f \in X$.

Examples:

- $E \subset \mathbb{R}^2$ measurable.
  $X = X_E = \{f \in L^2(\mathbb{R}^2, \mathbb{H}); \hat{f}(\xi) = 0 \text{ off } E\}$.

$$A_E(\xi) = \begin{pmatrix} \chi_E(\xi) & 0 \\ 0 & \chi_{-E}(\xi) \end{pmatrix}, \quad m(\xi) = \chi_E(\xi)$$

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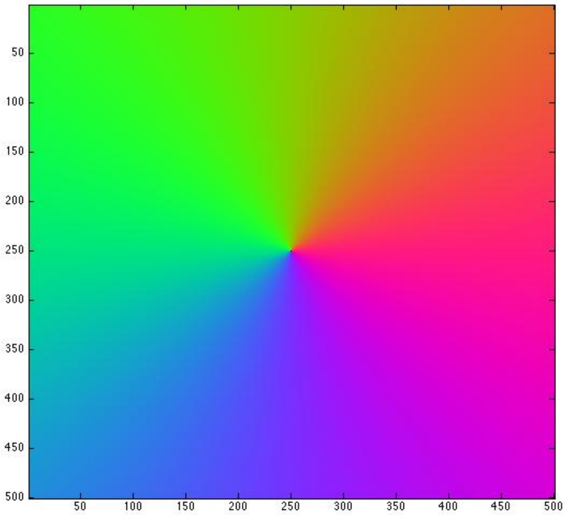
- $H^2_\pm(\mathbb{R}^d)$ the Hardy spaces of functions with "monogenic extensions" to $\mathbb{R}^d_\pm$.

$$A_\pm(\xi) = \frac{1}{2} \begin{pmatrix} 1 & \pm\xi/|\xi| \\ \mp\xi/|\xi| & 1 \end{pmatrix}; \quad m_\pm(\xi) = \frac{1}{2}\left(1 \pm \frac{\xi}{|\xi|}\right)$$

The Hilbert multiplier



Continuous wavelet transform

$\psi \in L^2(\mathbb{R}^2, \mathbb{R}_2)$, $\psi_t(x) = t^{-2}\psi(x/t)$, $\psi^*(x) = \overline{\psi(-x)}$.

Wavelet transform: $f \mapsto W_\psi f(x,t) = f * \psi_t^*(x)$

Calderón singular integral: $T_\psi f(x) = \int_0^\infty W_\psi f(\cdot, t) * \psi_t(x) \, \dfrac{dt}{t}$

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Theorem (Morris, H. (2012))

T_ψ bounded and invertible if and only if there exist constants $0 < A \leq B < \infty$ such that

$$A.I \leq \int_0^\infty [\hat{\psi}(t\xi)]^*[\hat{\psi}(t\xi)]\, \frac{dt}{t} \leq B.I$$

for a.e. ξ.

Quaternionic scaling functions in $L^2(\mathbb{R}^2, \mathbb{H})$

$\{h_k\} \in \ell^2(\mathbb{Z}^2, \mathbb{H})$ then $m_0(y) = \sum_{\ell \in \mathbb{Z}^2} e^{2\pi \ell \wedge y} h_k$

Theorem (H, Morris (2012))

$\{\varphi(x-\ell)\}_{\ell \in \mathbb{Z}^2}$ *orthonormal in* $L^2(\mathbb{R}^2, \mathbb{H})$ *if and only if*

$$\sum_{\ell \in \mathbb{Z}^2} [\hat{\varphi}(y+\ell)][\hat{\varphi}(y+\ell)]^* = I \qquad (1)$$

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φ *is self-similar if*

$$\frac{1}{4}\varphi\left(\frac{x}{2}\right) = \sum_{\ell\in\mathbb{Z}^2} \varphi(x-\ell)h_\ell \iff [\hat{\varphi}(2y)] = [\hat{\varphi}(y)][m_0(-y)] \tag{2}$$

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$$(1)+(2) \Longrightarrow [m_0(0)] = I \text{ and } \sum_{p\in P} [m_0(y+p)]^*[m_0(y+p)] = I$$

(QMF condition) with $P = \{0, (1/2, 0), (0, 1/2), (1/2, 1/2)\}$.

Quaternionic wavelets

$$[\hat{\psi}_j(2y)] = [m_j(y)][\hat{\varphi}(y)] \quad (1 \leq j \leq 3)$$

$$U(y) = \begin{pmatrix} [m_0(y)] & [m_1(y)] & [m_2(y)] & [m_3(y)] \\ [m_0(y+p_1)] & [m_1(y+p_1)] & [m_2(y+p_1)] & [m_3(y+p_1)] \\ [m_0(y+p_2)] & [m_1(y+p_2)] & [m_2(y+p_2)] & [m_3(y+p_2)] \\ [m_0(y+p_3)] & [m_1(y+p_3)] & [m_2(y+p_3)] & [m_3(y+p_3)] \end{pmatrix}$$

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Theorem (H, Morris (2012))

$\{2^j \psi_j(2^j - k);\ 1 \leq j \leq 3,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^2\}$ *o.n.b. for* $L^2(\mathbb{R}^2, \mathbb{H})$ *if and only if*

$$U(0) = I \text{ and } U(y)U(y)^* = I \text{ for a.e. } y$$

Wavelet basis construction

Scalar case, $d = 1$:

$$U(\xi) = \begin{pmatrix} m_0(\xi) & m_1(\xi) \\ m_0(\xi + 1/2) & m_1(\xi + 1/2) \end{pmatrix}$$

$$\varphi \longleftrightarrow m_0$$

$$\begin{aligned} m_0 \text{ a trig poly} \quad &\Longleftrightarrow \quad \{h_k\} \text{ a finite sequence} \\ &\Longleftrightarrow \quad \varphi \text{ compactly supported} \end{aligned}$$

Wavelet ψ: $\hat{\psi}(2\xi) = m_1(\xi)\hat{\varphi}(\xi)$. (φ, ψ) a mother-father wavelet pair if and only if

- $U(\xi)$ unitary for all ξ and
- $U(0) = I$

Wavelet basis construction

$$m_0, \;\; m_1 \text{ trig polys } \Rightarrow U(\xi) = \sum_{k=0}^{M-1} A_k e^{2\pi i k \xi}$$

$$\boxed{I = U(\xi)U(\xi)^* \iff \sum_{k=0}^{M-1-\ell} A_k A_{k+\ell}^* = \delta_\ell} \qquad (3)$$

Samples of $U(\xi)$: $U_\ell = U(\ell/M)$

$$U_\ell = \sum_{k=0}^{M-1} A_k e^{2\pi i k\ell/M} \Rightarrow A_k = \frac{1}{M} \sum_{\ell=0}^{M-1} U_\ell e^{-2\pi i \ell k/M} \qquad (4)$$

Wavelet basis construction

Proposition

$U(\xi)$ unitary for all ξ if and only if

$$\boxed{\sum_{n=0}^{M-1}\sum_{j=0}^{M-1} b_{nj}^{(\ell)} U_n^* U_j = M^2 \delta_\ell I \quad (0 \le \ell \le M-1)}$$

where $b_{nj}^{(\ell)} = e^{-2\pi i \ell j/M} \sum_{k=0}^{M-1-\ell} e^{2\pi i k(n-j)/M}$.

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where $b_{nj}^{(\ell)} = e^{-2\pi i \ell j/M} \sum_{k=0}^{M-1-\ell} e^{2\pi i k(n-j)/M}$.

Equivalently,

$$\boxed{\{U_k\}_{k=0}^{M-1} \text{ unitary and } U_k^* V_k = V_k^* U_k \quad (0 \le k \le M-1)}$$

where

$$V_k = \sum_{m \ne k} \frac{U_m}{e^{2\pi i (k-m)/M} - 1}$$

Reformulation

We want to find three M-tuple of matrices

$$\mathbf{U} = (U_0, U_1, \ldots, U_{M-1})$$
$$\mathbf{V} = (V_0, V_1, \ldots, V_{M-1})$$
$$\mathbf{W} = (W_0, W_1, \ldots, W_{M-1})$$

such that

(i) each U_n is unitary;

(ii) $V_n = \sum_{m \neq n} a_{m-n} U_m$; $\quad \left(a_m = \dfrac{1}{1 - e^{2\pi i m/M}} \right)$

(iii) $W_n = V_n^* U_n$

(iv) each W_n is self-adjoint

Minimization

$$F(\mathbf{U},\mathbf{V},\mathbf{W}) = \frac{1}{2}\sum_{n=0}^{M-1} \| V_n - \sum_{m\neq n} a_{mn} U_m \|^2 + \frac{1}{2}\sum_{n=0}^{M-1} \| W_n - V_n^* U_n \|^2$$

Then compute

$$\min_{\mathbf{U},\mathbf{V},\mathbf{W}} F(\mathbf{U},\mathbf{V},\mathbf{W})$$

subject to the constraints

(i) U_n unitary;

(ii) W_n self-adjoint

Other constraints

Consistency:

$$U(\xi+1/2) = \begin{pmatrix} m_0(\xi+1/2) & m_1(\xi+1/2) \\ m_0(\xi) & m_1(\xi) \end{pmatrix} = JU(\xi) \qquad (5)$$

where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Equivalently,

$$\boxed{U_{\ell+M/2} = JU_\ell \quad (0 \le \ell \le M/2-1)}$$

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Equivalently,

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Regularity: $m_0'(1/2) = m_0''(1/2) = \cdots = m_0^{(j)}(1/2) = 0$

$$\boxed{U^{(j)}(0) = \sum_{\ell=0}^{M-1} c_\ell^{(j)} U_\ell}$$

Thanks!